

Quark Number Susceptibility with Finite Quark Mass in Holographic QCD ^{*)}

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We study the effect of a finite quark mass on the quark number susceptibility in the framework of holographic QCD. We work in a bottom-up model with a deformed AdS black hole and D3/D7 model to calculate the quark number susceptibility at finite temperature with/without a finite quark chemical potential. As expected the finite quark mass suppresses the quark number susceptibility. We find that at high temperatures $T \geq 600$ MeV the quark number susceptibility of light quarks and heavy quarks are almost equal in the bottom-up model. This indicates that the heavy quark like charm contribution to thermodynamics of a QCD-like system may start to become significant at temperatures $T \sim 600$ MeV. In D3/D7 model, we focus on the competition between the quark chemical potential, which enhances the quark number susceptibility, and the quark mass that suppresses the susceptibility. We observe that depending on the relative values of the quark mass and the quark chemical potential, the quark number susceptibility shows a diverging or converging behavior. We also calculate the chiral susceptibility in D3/D7 model to support the observation made with the quark number susceptibility.

§1. Introduction

Fluctuations of conserved charges such as baryon number, electric charge, and strangeness are generally considered as useful probes for the structure of a thermal medium, quark gluon plasma (QGP), produced in heavy ion collisions. Quark number fluctuations are basic observables which can be obtained by taking derivatives of the grand canonical potential with respect to the quark chemical potential. The existence of a peak in the second order quark number fluctuation, or quark number susceptibility, near T_c is confirmed by recent lattice QCD calculations based on the Taylor expansion with respect to the quark (or baryon) chemical potential.^{1),2)} This implies the existence of the critical end point (CEP), at which the first order phase transition terminates in the (μ_q, T) plane of the QCD phase diagram (see³⁾ for a review). There have been many studies to calculate the quark number susceptibility and higher order quark number fluctuations in various model studies^{4)–11)} and lattice

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simulations.^{12)–16)} In,^{17)–19)} the quark number susceptibility at finite temperature with finite chemical potential is studied in a holographic QCD model.

The AdS/CFT correspondence^{20)–22)} is a powerful tool to study strongly coupled gauge theories. Using this correspondence, we can obtain physical quantities in gauge theories from gravity side. We can also discuss finite temperature systems from this gauge/gravity correspondence. In the hydrodynamic limit i.e. low frequency limit, we can calculate correlation functions.^{23),24)} The most fruitful hydrodynamic result is the observation of the “universal” bound of the ratio of the shear viscosity to entropy density.²⁵⁾ Although the correspondence between QCD and gravity theory is not known, we can obtain much insights by using this correspondence.

In this paper we study the effect of a finite quark mass on the quark number susceptibility in holographic QCD. We consider light, strange and heavy (charm) quarks. In the previous studies,^{17),18)} the mass of quark is zero, and so the quark number susceptibility with two massless flavor was calculated in a QCD-like system. We now extend the previous work by considering non-zero quark masses. Our basic motivation for the charm quark is an observation made in perturbative QCD analysis on the role of quark mass in QCD thermodynamics. In,²⁶⁾ it is claimed that unexpectedly, the charm quark plays some role in the QGP at relatively low temperatures $T \sim 350$ MeV. Since this analysis is based on the perturbation, it should be important to check the claim in a non-perturbative study. To explore the effect of a finite quark mass at finite temperature and/or density, we work with two models. The first one is a bottom-up type model based on the deformed AdS black hole due to finite quark mass obtained in.²⁷⁾ Then we consider D3/D7 model with a finite chemical potential studied in.^{28),29)}

This paper is organized as follows: After introducing the quark number susceptibility, we calculate it in the deformed AdS black hole background in Section 2. The hydrodynamic and thermodynamic analysis are given in the section. In Section 3, we discuss the quark number susceptibility in D3/D7 system. We summarize the result in the final section. The procedure to obtain the retarded Green function is briefly given in the Appendix A. In Appendix B, we consider the chiral susceptibility in D3/D7 model, though the chiral symmetry in D3/D7 is only $U(1)_A$ not that of QCD, $SU(N_f)_L \times SU(N_f)_R$.

§2. Quark number susceptibility in deformed AdS black hole

The quark number susceptibility was proposed as a probe of the QCD chiral phase transition at zero chemical potential,^{4),5)}

$$\chi_q = \frac{\partial n_q}{\partial \mu}. \quad (2.1)$$

Later it has been shown that the quark number susceptibility can be rewritten in terms of the retarded Green function through the fluctuation-dissipation theorem,⁶⁾

$$\chi_q = - \lim_{k \rightarrow 0} \text{Re} \left(G_{t \ t}(\omega = 0, k) \right), \quad (2.2)$$

where $G_{\mu \ \nu}(\omega, k)$ is the retarded Green function of the vector-vector correlations.

Now, we calculate the quark number susceptibility in the deformed AdS black hole background. We do this in two different ways that should be equivalent each other. We first use the equation (2.2) to conduct hydrodynamic analysis and then calculate the quark number susceptibility using the thermodynamic relation in the equation (2.1).

The background we shall discuss is AdS black hole with back-reactions coming from flavors,²⁷⁾

$$(ds)^2 = e^{-2K(z)} \left(-f(z)(dt)^2 + (d\vec{x})^2 \right) + \frac{(dz)^2}{z^2 f(z)}, \quad (2.3)$$

with

$$K(z) = \log z + \frac{1}{4} \left(\frac{z}{z_Q} \right)^2, \quad (2.4a)$$

$$f(z) = 1 - 2 \left(\frac{z_Q}{z_h} \right)^4 \left\{ 1 - \left(1 - \left(\frac{z}{z_Q} \right)^2 \right) e^{(z/z_Q)^2} \right\}. \quad (2.4b)$$

The parameter z_Q is defined as

$$z_Q^2 = \frac{6}{\kappa^2 N_f M_q^2}. \quad (2.5)$$

M_q represents the quark mass and $\kappa^2 = 8\pi G_5$ with $1/G_5 = 32N_c^2/(\pi L^3)$ where L is the AdS radius. In²⁷⁾ there were two deformations to be discussed, deformed AdS and deformed AdS black hole. In the present work we only consider one of them, i.e. the deformed AdS black hole background. Each of these two backgrounds is reasonable with given potentials*). Note that in²⁷⁾ N_f counts the number of massive quarks only. For instance, if we are interested in a 2+1 flavor system, we take $N_f = 1$ in the equation (2.5), neglecting the back-reaction from the two flavors whose masses are zero or very small. Taking the limit $M_q \rightarrow 0$, the metric (2.3) is reduced to the AdS black hole

$$(ds)^2 = \frac{1}{z^2} \left(-f(z)(dt)^2 + (d\vec{x})^2 \right) + \frac{(dz)^2}{z^2 f(z)},$$

with

$$f(z) = 1 - \left(\frac{z}{z_h} \right)^4,$$

where z_h implies the location of horizon of the original AdS black hole. There exists a horizon z_H which gives zero of the function (2.4b)**). Then the Hawking temperature

*) In²⁷⁾ the authors discussed the Hawking-Page type transition^{30), 31)} which is the dual to the confinement/deconfinement transition using these two deformed backgrounds. However, since the bulk potentials for each background are not the same, the Hawking-Page type analysis in²⁷⁾ needs to be improved.

**) We may refer the position of horizon as

$$z_H = z_Q \sqrt{1 + \text{ProductLog} \left(\frac{(z_h/z_Q)^4 - 2}{2e} \right)},$$

where $\text{ProductLog}(a)$ stands for a solution for x in the equation $xe^x = a$.

is given by

$$T = \frac{1}{\pi z_H} \left(\frac{z_H}{z_h} \right)^4 e^{\frac{3}{4} \left(\frac{z_H}{z_Q} \right)^2}. \quad (2.6)$$

We note that for small M_q , z_H becomes z_h . Using the background (2.3), we assume M_q is not large within the small mass deformation. Since the background (2.3) is constructed from the bottom-up type approach, we could not find any restrictions for the finite value of M_q .

Let us now consider 5D $U(1)$ gauge field which is dual to 4D quark number current on the background (2.3). We then calculate the quark number susceptibility by using the AdS/CFT correspondence. Since in this background there is no background charge which could be interpreted as the chemical potential in the boundary theory, we here discuss the mass dependence of the quark number susceptibility.

We start by introducing a dimensionless coordinate $u \equiv (z/z_H)^2$ which is normalized by the horizon. In this coordinate system, the horizon and the boundary are located at $u = 1$ and $u = 0$, respectively. The metric (2.3) is rewritten as

$$(ds)^2 = e^{-2K(u)} \left(-f(u)(dt)^2 + (d\vec{x})^2 \right) + \frac{(du)^2}{4u^2 f(u)}, \quad (2.7)$$

with

$$K(u) = \frac{1}{2} \log(z_H^2 u) + \frac{1}{4} \left(\frac{z_H}{z_Q} \right)^2 u, \quad (2.8a)$$

$$\begin{aligned} f(u) &= 1 - 2 \left(\frac{z_Q}{z_h} \right)^4 \left\{ 1 - \left(1 - \left(\frac{z_H}{z_Q} \right)^2 u \right) e^{\left(\frac{z_H}{z_Q} \right)^2 u} \right\} \\ &= \frac{1}{1 - \left(1 - \left(\frac{z_H}{z_Q} \right)^2 \right) e^{\left(\frac{z_H}{z_Q} \right)^2}} \left\{ \left(1 - \left(\frac{z_H}{z_Q} \right)^2 u \right) e^{\left(\frac{z_H}{z_Q} \right)^2 u} \right. \\ &\quad \left. - \left(1 - \left(\frac{z_H}{z_Q} \right)^2 \right) e^{\left(\frac{z_H}{z_Q} \right)^2} \right\}. \end{aligned} \quad (2.8b)$$

The action for the $U(1)$ gauge field is

$$S = -\frac{1}{4g_5^2} \int d^5x \sqrt{-g} F_{mn} F^{mn}, \quad (2.9)$$

where g_5 is the 5D gauge coupling constant. We shall work in $A_u(x) = 0$ gauge and use the Fourier decomposition

$$A_\mu(t, z, u) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + ikz} A_\mu(\omega, k, u), \quad (2.10)$$

where we choose the spacial momenta which are along the z -direction. Variations of the action with respect to $A_t(u)$ and $A_u(u)$ give equations of motion

$$0 = \left(e^{-\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 u} A'_t(u) \right)' - \frac{z_H^2}{4u f(u)} \left(k^2 A_t(u) + \omega k A_z(u) \right), \quad (2.11a)$$

$$0 = \omega A'_t(u) + k f(u) A'_z(u), \quad (2.11b)$$

where the prime stands for the derivative with respect to u . The equation of motion for $A_z(u)$ can be derived from the equations (2.11a) and (2.11b). For $A_x(u)$ and $A_y(u)$, one can obtain decoupled equations of motion. Since we are interested in the time-time component of the retarded Green function to calculate the quark number susceptibility (2.2), we will not consider $A_x(u)$ and $A_y(u)$ hereafter.

From the equations (2.11a) and (2.11b), we obtain an equation for $A_t(u)$,

$$0 = \left(u f(u) e^{-\left(\frac{z_H}{z_Q}\right)^2 u} A_t''(u) \right)' - \left\{ \frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 e^{-\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 u} \left(u f(u) e^{-\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 u} \right)' - \frac{z_H^2}{4} e^{-\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 u} \left(\frac{\omega^2}{f(u)} - k^2 \right) \right\} A_t'(u). \quad (2.12)$$

We can make the equation more convenient form by changing the variable,

$$A_t'(u) = e^{\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 u} X(u). \quad (2.13)$$

Then the equation (2.12) can be rewritten as

$$0 = \left(u f(u) X'(u) \right)' + \frac{z_H^2}{4} e^{\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 u} \left(\frac{\omega^2}{f(u)} - k^2 \right) X(u). \quad (2.14)$$

Let us proceed to solve the equation of motion (2.14). This is an ordinary second order differential equation with a regular singularity at the horizon $u = 1$. Writing a solution as $X(u) = (1 - u)^\nu F(u)$ where $F(u)$ is a regular function at the horizon, we fix the constant ν by imposing the incoming wave condition,

$$\nu = -i \frac{\omega}{4\pi T}, \quad (2.15)$$

where T is the Hawking temperature (2.6).

Now we shall solve the equation of motion in the hydrodynamic regime i.e. small ω and k compared with the temperature T . In this regime, we could expand the function $F(u)$ as

$$F(u) = F_0(u) + \omega F_\omega(u) + k^2 F_{k^2}(u) + \mathcal{O}(\omega^2, \omega k^2). \quad (2.16)$$

Since we are interested in the quark number susceptibility defined by (2.2), we only need solutions for $F_0(u)$ and $F_{k^2}(u)$. Relevant equations are read off as

$$0 = \left(u f(u) F_0'(u) \right)', \quad (2.17a)$$

$$0 = \left(u f(u) F_{k^2}'(u) \right)' - \frac{z_H^2}{4} e^{\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 u} F_0(u). \quad (2.17b)$$

In the equation (2.17a), avoiding the singularity at the horizon, we fix the function $F_0(u)$ as a constant C which is determined later,

$$F_0(u) = C. \quad (2.18)$$

Using this solution, the equation (2.17b) leads

$$F'_{k^2}(u) = C \frac{z_H^2}{4uf(u)} \int_1^u du' e^{\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2 u'} = C \frac{z_H^2}{4uf(u)} 2\left(\frac{z_Q}{z_H}\right)^2 \left(e^{\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2 u} - e^{\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2} \right), \quad (2.19)$$

where we have imposed the regularity at the horizon. We insert the solutions above into the equation (2.11a) and obtain

$$\frac{z_H^2}{4uf(u)} \left(k^2 A_t(u) + \omega k A_z(u) \right) = X'(u) = k^2 F'_{k^2}(u) + \mathcal{O}(\omega). \quad (2.20)$$

Taking the definition of the quark number susceptibility (2.2) into account, the equation (2.20) is enough to determine $A_t(u)$,

$$\begin{aligned} A_t(u) &= 2C \left(\frac{z_Q}{z_H} \right)^2 \left(e^{\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2 u} - e^{\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2} \right) \\ &= \frac{A_t^0}{1 - e^{\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2}} \left(e^{\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2 u} - e^{\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2} \right), \end{aligned} \quad (2.21)$$

where we have defined the boundary variable as $A_t^0 \equiv A_t(u)|_{u \rightarrow 0}$ to fix the constant C .

Let us evaluate the quark number susceptibility. The on-shell action to be needed is estimated as

$$S = \frac{1}{g_5^2 z_H^2} \int \frac{d^4 k}{(2\pi)^4} e^{-\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2 u} A_t(-k) A'_t(k) \Big|_{u=1}^{u=0}. \quad (2.22)$$

Following the procedure given in Appendix A, we can read off the two point function $G_{tt}(\omega, k)$. We then conclude that the quark number susceptibility is given as

$$\chi_q = \frac{1}{g_5^2 z_Q^2 \left(e^{\frac{1}{2}\left(\frac{z_H}{z_Q}\right)^2} - 1 \right)}. \quad (2.23)$$

In Fig.1, we plot χ_q/T^2 as a function of T with varying the quark mass M_q^* . Note here that the quark mass in our model can be different from that in QCD (or in the chiral Lagrangian) by a constant factor. For instance, in the hard wall model the factor is $\sqrt{3}$, which is obtained by matching the scalar two-point correlators calculated from QCD and the hard wall model:³²⁾ $M_q^{\text{ChPT}} = M_q/3$. If we assume that this is valid in our case, we have to divide the quark mass in our model by the factor $\sqrt{3}$ when we compare our results with those from QCD or chiral perturbation theory (ChPT). In Fig.1, the vertical dashed line is for a typical deconfinement temperature with no quark mass effect in the hard wall model.³³⁾ If we assume that the deconfinement temperature in our approach is roughly equal to that in

^{*)} We are interested in the qualitative tendency of the mass dependence to the quark number susceptibility. For the illustration purpose, we vary M_q from 100 to 2000 MeV.

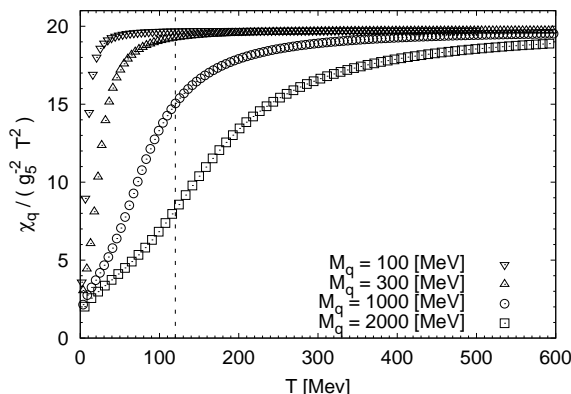


Fig. 1. The quark number susceptibility as a function of the temperature for various quark masses. The vertical dashed line denotes the deconfinement temperature in the hard wall model.

the hard wall model^{*)}, our results on the left of this line are just for illustration and might be replaced by those with the thermal AdS background as in.^{17),18)} As expected the finite quark mass suppresses the quark number susceptibility. We find that at high temperatures $T \geq 600$ MeV the quark number susceptibility of light quarks and heavy quarks are almost equal. This indicates that the heavy quark like charm contribution to thermodynamics of a QCD-like system may start to become significant at temperatures $T \sim 600$ MeV. Though it might not mean much, we compare our number $T \sim 600$ with that from lattice QCD. In,³⁴⁾ it is shown that the fluctuations of the charm quark number is comparable to those of the light quark sector at high temperatures and claim that at temperatures $T \sim 800$ MeV the contribution of charm quark to QCD thermodynamics is significant.

Before closing this section, we briefly demonstrate the thermodynamic analysis on this system. In the background (2.7), we turn on the temporal component of gauge field which could introduce the finite density in the boundary gauge theory. We work in the grand canonical ensemble. In the AdS/CFT correspondence, we could identify the grand potential $\Omega(T, \mu)$ as

$$\Omega = TS_{\text{on-shell}}, \quad (2.24)$$

where $S_{\text{on-shell}}$ is the 5D bulk on-shell action evaluated in Euclidean spacetime. As in the usual imaginary time analysis, we analytically continue to Euclidean spacetime and compactify the time direction with the period $1/T$. Assuming that the gauge field depends only on the radial coordinate u , the equation of motion leads

$$A'_t(u) = -d e^{\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 u}, \quad (2.25)$$

where the integration constant d (> 0) will be proportional to the quark number

^{*)} We should consider the Hawking-Page type transition^{30),31)} in the current setup. Since the analysis for this confinement/deconfinement in²⁷⁾ needs to be improved as we explained, we here take the hard wall model as the closest existing one.

density. Then the grand potential (2.24) can be estimated as

$$\Omega = \frac{d^2 V_3}{g_5^2 z_H^2} \int_1^0 du e^{\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2 u} = \frac{2d^2 V_3 z_Q^2}{g_5^2 z_H^4} \left(1 - e^{\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2} \right), \quad (2.26)$$

where V_3 is 3D volume. The chemical potential is defined by

$$\mu = \int_1^0 du A'_t(u) = -\frac{2dz_Q^2}{z_H^2} \left(1 - e^{\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2} \right). \quad (2.27)$$

Using these relations, we obtain

$$\Omega = \frac{V_3 \mu^2}{2g_5^2 z_Q^2 \left(1 - e^{\frac{1}{2} \left(\frac{z_H}{z_Q} \right)^2} \right)}. \quad (2.28)$$

Since the quark number density is defined by

$$n_q = -\frac{1}{V_3} \frac{\partial \Omega}{\partial \mu} \left(= \frac{2}{g_5^2 z_H^2} d \right), \quad (2.29)$$

one can easily confirm that the grand potential exactly gives the quark number susceptibility obtained in (2.23). As it should be, the hydrodynamic and thermodynamic analysis give the same result.

§3. Quark number susceptibility in D3/D7 system

In this section, we study effects of the chemical potential and the quark mass for the quark number susceptibility at finite temperature in a D3/D7 system. Following,^{28),29)} we consider the “black hole embedding” of the probe D7-branes in the D3-brane background in which we could introduce the finite quark mass together with the chemical potential. In this section we follow the procedure developed in²⁹⁾ and then extract relevant parts to discuss the quark number susceptibility.

We start by introducing N_c D3-branes. The AdS/CFT correspondence provides the dual descriptions of $\mathcal{N} = 4$ $SU(N_c)$ SYM as the type IIB string theory on $AdS_5 \times S^5$ with the identifications $L^4/l_s^4 = 2g_{YM}^2 N_c = 2\lambda$ and $g_s = g_{YM}^2/(2\pi)$. Here L , $l_s = \sqrt{\alpha'}$ and λ are the radius in AdS_5 and S^5 , the string length and 't Hooft coupling, respectively. Taking the large N_c and the large 't Hooft coupling limit, the string description reduces to that for the supergravity. The finite temperature system in the deconfined phase can be realized by using a corresponding black hole geometry in the Euclidean signature:

$$(ds)^2 = \frac{u^2}{L^2} \left(f_0 (dt)^2 + (d\vec{x})^2 \right) + \frac{L^2}{u^2} \left(\frac{(du)^2}{f_0} + u^2 d\Omega_5^2 \right), \quad (3.1)$$

with

$$f_0(u) = 1 - \frac{u_0^4}{u^4},$$

where u_0 implies the location of horizon. The periodic thermal identification $t \simeq t + 1/T$ leads the Hawking temperature

$$T = \frac{u_0}{\pi L^2}. \quad (3.2)$$

Following the paper,³⁷⁾ we introduce new coordinate ϱ through

$$\varrho^2 = u^2 + \sqrt{u^4 - u_0^4}, \quad (3.3)$$

and rewrite the metric (3.1) as

$$(ds)^2 = \frac{1}{2} \frac{\varrho^2}{L^2} \left(\frac{f^2}{\tilde{f}} (dt)^2 + \tilde{f} (d\vec{x})^2 \right) + \frac{L^2}{\varrho^2} \left((d\varrho)^2 + \varrho^2 d\Omega_5^2 \right), \quad (3.4)$$

with

$$f(\varrho) = 1 - \frac{u_0^4}{\varrho^4}, \quad \tilde{f}(\varrho) = 1 + \frac{u_0^4}{\varrho^4}.$$

We work with the dimensionless coordinate $\rho \equiv \varrho/u_0$ in the rest of the paper. We refer to the horizon as $\rho = 1$ and the AdS boundary as $\rho (= \sqrt{2}u/u_0) \rightarrow \infty$.

In order to introduce the fundamental matters, we may consider flavor branes,³⁵⁾ which provide flavor gauge fields in the world volume. Here we put N_f D7-branes with the following intersection

	0	1	2	3	4	5	6	7	8	9
D3:	×	×	×	×						
D7:	×	×	×	×	×	×	×	×		

(3.5)

where D7-branes are wrapping on S^3 of S^5 . The additional degrees of freedom which we would like to discuss are generated by open string oscillations between D3 and D7-branes described by $\mathcal{N} = 2$ hypermultiplet. The fermions in this multiplet could be interpreted as the analog of quarks in QCD. Taking the probe approximation $N_f \ll N_c$, we could consider the overall dynamics of N_f D7-branes which can be described by the DBI action in the 10D background (3.4),

$$S = N_f T_7 \int d^8 \sigma \, e^{-\phi} \sqrt{\det(g_{mn} + 2\pi\alpha' F_{mn})}, \quad (3.6)$$

where $g_{mn}(\sigma)$ and $F_{mn}(\sigma)$ are the induced metric and the field strength of the world volume gauge field, respectively. The tension of D7-branes and the dilaton field are given by $T_7 = 1/((2\pi)^7 l_s^8)$ and $e^\phi = g_s$, respectively. It is convenient to divide the transverse 6D part to D3-branes in (3.4) into two parts i.e. 4D and 2D whose coordinates are given by spherical (r, Ω_3) and polar (R, φ) coordinates, respectively,

$$\begin{aligned} (d\rho)^2 + \rho^2 d\Omega_5^2 &= (dr)^2 + r^2 d\Omega_3^2 + (dR)^2 + R^2 (d\varphi)^2 \\ &= (d\rho)^2 + \rho^2 \left((d\theta)^2 + \sin^2 \theta d\Omega_3^2 + \cos^2 \theta (d\varphi)^2 \right), \end{aligned} \quad (3.7)$$

where $r = \rho \sin \theta$, $R = \rho \cos \theta$ with $0 \leq \theta \leq \pi/2$ and $\rho^2 = r^2 + R^2$. By construction, we consider the case where the world volume coordinates of D7-branes are given in

the static gauge i.e. $\sigma^m \equiv (t, \vec{x}, \rho, \Omega_3)$. Due to the symmetries for the translation in (t, \vec{x}) and the rotation in (ρ, Ω_3) , the embedding of the D7-branes could only depend on the radial coordinate ρ . Since the rotational symmetry in (R, φ) allows to be $\varphi = 0$, the embedding might be characterized by $\chi(\rho) \equiv \cos \theta$ through $\theta(\rho)$ which is the angle between two spaces (r, Ω_3) and (R, φ) . The asymptotic value of the distance between D3 and D7-branes which is given by $R(\rho)$ for large ρ provides the quark mass M_q . The induced metric on the D7-branes is given by

$$(\text{ds})_{\text{D7}}^2 = L^2 \left\{ \frac{\pi^2 T^2}{2} \rho^2 \left(\frac{f^2}{\tilde{f}} (\text{d}t)^2 + \tilde{f} (\text{d}\vec{x})^2 \right) + \frac{1}{\rho^2} \left(\frac{1 - \chi^2 + \rho^2 \dot{\chi}^2}{1 - \chi^2} \right) (\text{d}\rho)^2 + (1 - \chi^2) \text{d}\Omega_3^2 \right\}, \quad (3.8)$$

where the dot stands for the derivative with respect to ρ . We also introduce the non-dynamical temporal component of the gauge field $A_t(\rho)$ which leads the chemical potential and the density at the AdS boundary.

By using the induced metric (3.8) and the gauge potential $A_t(\rho)$, the DBI action (3.6) now becomes

$$S_0 = \frac{\lambda N_c N_f T^3}{32} V_3 \int \text{d}\rho \, \rho^3 \tilde{f} (1 - \chi^2) \sqrt{f^2 (1 - \chi^2 + \rho^2 \dot{\chi}^2) - 2\tilde{f} (1 - \chi^2) \dot{\tilde{A}}_t^2}, \quad (3.9)$$

where we have defined $\tilde{A}_t(\rho) \equiv 2\pi\alpha' A_t(\rho)/u_0$. Since there exist no $\tilde{A}_t(\rho)$ terms in the action, the equation of motion for $\tilde{A}_t(\rho)$ can be reduced to the following form with an integration constant \tilde{d} ,

$$\tilde{d} \equiv \frac{\rho^3 \tilde{f}^2 (1 - \chi^2)^2 \dot{\tilde{A}}_t}{2 \sqrt{f^2 (1 - \chi^2 + \rho^2 \dot{\chi}^2) - 2\tilde{f} (1 - \chi^2) \dot{\tilde{A}}_t^2}}. \quad (3.10)$$

The equation of motion for $\chi(\rho)$ is given as

$$0 = \partial_\rho \left\{ \frac{\rho^5 f \tilde{f} (1 - \chi^2) \dot{\chi}}{\sqrt{1 - \chi^2 + \rho^2 \dot{\chi}^2}} \left(1 + \frac{8\tilde{d}^2}{\rho^6 \tilde{f}^3 (1 - \chi^2)^3} \right)^{1/2} \right\} + \frac{\rho^3 f \tilde{f} \chi}{\sqrt{1 - \chi^2 + \rho^2 \dot{\chi}^2}} \left\{ \left(3(1 - \chi^2) + 2\rho^2 \dot{\chi}^2 \right) \left(1 + \frac{8\tilde{d}^2}{\rho^6 \tilde{f}^3 (1 - \chi^2)^3} \right)^{1/2} - \frac{24\tilde{d}^2 (1 - \chi^2 + \rho^2 \dot{\chi}^2)}{\rho^6 \tilde{f}^3 (1 - \chi^2)^3} \left(1 + \frac{8\tilde{d}^2}{\rho^6 \tilde{f}^3 (1 - \chi^2)^3} \right)^{-1/2} \right\}, \quad (3.11)$$

where we have eliminated the gauge field $\tilde{A}_t(\rho)$ by using the relation (3.10). Near the boundary, asymptotic solutions of the equations of motion (3.10) and (3.11) behave as

$$\tilde{A}_t(\rho) = \tilde{\mu} - \frac{\tilde{d}}{\rho^2} + \cdots, \quad (3.12a)$$

$$\chi(\rho) = \frac{m}{\rho} + \frac{c}{\rho^3} + \dots \quad (3.12b)$$

The chemical potential μ can be defined as the boundary value of $A_t(\rho)$, while the quark mass M_q can be estimated through the asymptotic value of the separation of D3 and D7-branes i.e. $M_q = \rho\chi(\rho)/(2\pi\alpha')$ at the AdS boundary. Taking the rescaling of the gauge field \tilde{A}_t and the coordinate ρ into account, the integration constants $\tilde{\mu}$ and m can be related with these values,

$$\tilde{\mu} = \frac{2\pi\alpha'}{u_0}\mu = \sqrt{\frac{2}{\lambda}}\frac{\mu}{T}, \quad (3.13a)$$

$$m = 2\pi\alpha'\frac{\sqrt{2}}{u_0}M_q = \frac{2}{\sqrt{\lambda}}\frac{M_q}{T}. \quad (3.13b)$$

As we will estimate below^{*)}, the remaining constants \tilde{d} and c would be proportional to the quark number density n_q and the quark condensate $\langle\bar{\psi}\psi\rangle$, respectively.

We have to solve the equations of motion (3.10) and (3.11) numerically to determine the embedding and the property of the gauge field. Here we restrict to the black hole embedding in which the D7-branes touch the horizon, since this might be thermodynamically favored configuration in the system with finite density. We impose boundary conditions at the horizon as $\dot{\chi}(1) = 0$ and $\tilde{A}_t(1) = 0$ to remove singularities and $\chi(1) = \chi_0$. We fix m and $\tilde{\mu}$ which depend on χ_0 and \tilde{d} by matching the numerical solutions with the asymptotic forms at the boundary.

We now consider the on-shell action which is related with the partition function Z of the field theory in the context of AdS/CFT correspondence:

$$Z = e^{-S_{\text{on-shell}}}. \quad (3.14)$$

However the on-shell action in this system contains UV divergences. It is well-known that one can prepare local boundary counter terms for probe D-branes in AdS spacetime by applying the holographic renormalization.³⁶⁾ In the D3/D7 system, taking the asymptotic solution (3.12b) into account, the relevant boundary counter terms take the form,³⁷⁾

$$S_{\text{ct}} = \frac{\lambda N_c N_f T^3}{32} V_3 \left\{ -\frac{1}{4} \left((\rho_{\text{max}}^2 - m^2)^2 - 4mc \right) \right\}, \quad (3.15)$$

where ρ_{max} is the cut-off for UV divergences which may go to infinity after precise calculations. It should be noticed that there exist finite contributions in the counter terms. Together with these counter terms, we could obtain the regularized action

$$\begin{aligned} S &= S_0 + S_{\text{ct}} \\ &= \frac{\lambda N_c N_f T^3}{32} V_3 \left\{ \int_1^\infty d\rho \left(\rho^3 \tilde{f}(1 - \chi^2) \sqrt{f^2(1 - \chi^2 + \rho^2 \dot{\chi}^2) - 2\tilde{f}(1 - \chi^2) \dot{\tilde{A}}_t^2} \right. \right. \\ &\quad \left. \left. - \rho^3 + m^2 \rho \right) - \frac{1}{4} \left((m^2 - 1)^2 - 4mc \right) \right\}. \end{aligned} \quad (3.16)$$

^{*)} In the dimension three operator, we neglect contributions from squarks in the hypermultiplet.

Since we are interested in the black hole embedding, the integration supports from the horizon to the AdS boundary. Evaluating the on-shell action for (3.16) which is reduced to boundary values through the equation of motion, we could observe that the quark condensate is proportional to the integration constant c ,

$$\langle \bar{\psi}\psi \rangle = -\frac{T}{V_3} \frac{\partial}{\partial M_q} \log Z = -\frac{1}{8} \sqrt{\lambda} N_c N_f T^3 c, \quad (3.17)$$

where we have used the asymptotic solutions (3.12a) and (3.12b) and the relation (3.13b). As we did in the previous section, we could identify the grand potential $\Omega(\mu, T)$ as

$$\Omega = -T \log Z = T S_{\text{on-shell}}. \quad (3.18)$$

The quark number density can be also calculated through the on-shell evaluation,

$$n_q = -\frac{1}{V_3} \frac{\partial \Omega}{\partial \mu} = \frac{1}{4} \sqrt{\frac{\lambda}{2}} N_c N_f T^3 \tilde{d}. \quad (3.19)$$

Let us now calculate the quark number susceptibility numerically using (2.1) and (3.19). In Fig.2, we plot our results with $a = \tilde{\mu}/m \sim \mu/M_q$. Since $1/m \sim T/M_q$, we can consider the horizontal axis of the figure as the temperature with the choice of M_q fixed. For very small a ($\ll 1$) or $M_q \gg \mu/\sqrt{2}$, the quark mass is dominating the quark

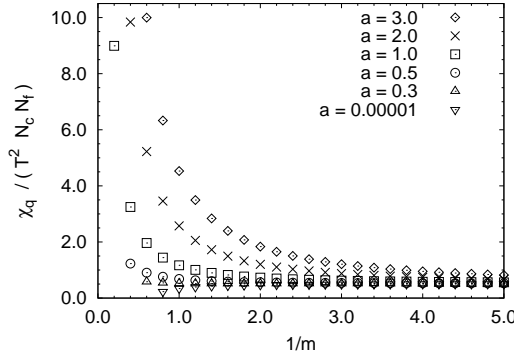


Fig. 2. The quark number susceptibility in the D3/D7 model as a function of $1/m$. Here $a = \tilde{\mu}/m = \mu/(\sqrt{2}M_q)$.

number susceptibility and suppresses it at low temperature, while in the opposite case the chemical potential enhances it. Note that the confinement/deconfinement transition temperature in the D3/D7 model is zero, since there is no dimensionful parameter which could characterize the critical temperature as in the Hawking-Page transition.^{30),31)} Indeed, within the large N_c approximation, one can observe the entropy is zero for $T = 0$, while for given finite T that is proportional to $N_c^2 T^3$ whose N_c^2 dependence indicates the deconfinement of the color degrees of freedom.³¹⁾ In addition, if one would like to consider mesons which correspond to the fluctuations of D7 flavor branes, one could observe their discrete mass spectra at $T = 0$.³⁸⁾ Another cautionary remark is that in the QCD phase diagram, there exists a CEP

at a single value of the chemical potential, while Fig.2 may imply multiple points in the sense that the quark number susceptibility is diverging for many different values of the chemical potential with fixed quark mass. At this moment, we have no clear understanding how to remove multiple CEPs. We may hope that some large N_c corrections could resolve this problem. Without resolving this issue, we may not be able to address single CEP in the QCD phase diagram within D3/D7 model.

Now we discuss implications of our findings in QCD at high temperature. We first spell out some limitations of our approach. As well known, a generic problem with this kind of holographic QCD is that the results from holographic QCD are mostly large N_c leading ones. Though there are some studies to include the corrections, most of them include some subset of the corrections, not all of them in a consistent way. Apart from this generic plague, the D3/D7 model we used here shows that deconfinement temperature T_c is zero. In addition, we assumed an exact isospin symmetry to use the abelian DBI action for the N_f probe branes, and so our approach could not study N_f -dependent nature of QCD phase transition. Unlike QCD, the temperature T , the chemical potential μ , and the quark mass M_q are not all independent, and only two of them or ratios of them are independent, which is the relic of conformal nature of our background metric. With the above-mentioned in mind, we comment on QCD, more precisely a QCD-like system, at finite temperature based on Fig.2.

For small a ($\ll 1$) or $M_q \gg \mu/\sqrt{2}$, the quark number susceptibility does not show any diverging behavior as the temperature approaches T_c , indicating that if the quark mass is big enough compared to the value of the quark chemical potential, the transition to deconfined phase is a smooth crossover rather than first or second order phase transition. In the opposite case $a > 1$, it shows a rapid jump-up, coming close to T_c from high temperature. This implies that as long as the quark chemical potential is a few times larger than the quark mass, the transition would be second order. In Appendix B, we calculate the chiral susceptibility in the D3/D7 model, see Fig.3. We observe that it shows a similar behavior with the quark number susceptibility in Fig.2.

§4. Summary

We studied the effect of a finite quark mass on the quark number susceptibility in the framework of holographic QCD. We first considered the bottom-up model with the deformed AdS black hole that includes the back-reaction of the quark mass. As expected the finite quark mass suppressed the quark number susceptibility. We observed that at high temperatures $T \geq 600$ MeV the quark number susceptibility of light quarks and heavy quarks are almost equal, indicating that the heavy quark like charm contribution to thermodynamics of a QCD-like system may start to become significant at such temperatures.

We then moved to the D3/D7 model to calculate the quark number susceptibility at finite temperature with a finite quark chemical potential. We studied the competition between the quark chemical potential, which enhances the quark number susceptibility, and the quark mass that suppresses the susceptibility. We observed

that depending on the relative values of the quark mass and the quark chemical potential, the quark number susceptibility shows diverging or converging behavior. With a caution that our approach is still not that close to a realistic system like quark-gluon plasma, we discussed physical implication of our results in a QCD-like system at the end of section 3.

We also calculated the chiral susceptibility in D3/D7 model in Appendix B to support the observation made with the quark number susceptibility.

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Appendix A

—— Minkowskian correlators in AdS/CFT correspondence ——

In this appendix, we briefly summarize the prescription for the Minkowskian correlator in AdS/CFT correspondence. We here follow the prescription proposed in.²³⁾ We work on the following 5D background,

$$(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{uu}(du)^2, \quad (\text{A}\cdot 1)$$

where x^μ and u are the 4D and radial coordinates, respectively. We refer the boundary at $u = 0$ and the horizon at $u = 1$. Let us consider a solution of an equation of motion in this 5D background. Suppose a solution of an equation of motion is given by

$$\phi(u, x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} f_k(u) \phi^0(k), \quad (\text{A}\cdot 2)$$

where $f_k(u)$ is normalized such that $f_k(0) = 1$ at the boundary. After putting the equation of motion back into the action, the on shell action might be reduced to surface terms

$$S[\phi^0] = \int \frac{d^4k}{(2\pi)^4} \phi^0(-k) \mathcal{G}(k, u) \phi^0(k) \Big|_{u=1}^{u=0}. \quad (\text{A}\cdot 3)$$

Here, the function $\mathcal{G}(k, u)$ can be written in terms of $f_{\pm k}(u)$ and $\partial_u f_{\pm k}(u)$. Accommodating Gubser-Klebanov-Polyakov/Witten relation^{21), 22)} to Minkowski space-

time, Son and Starinets proposed the formula to get the retarded Green functions,

$$G(k) = 2\mathcal{G}(k, u) \Big|_{u=0}, \quad (\text{A}\cdot 4)$$

where the incoming boundary condition at the horizon is imposed. In this paper, we consider correlators of $U(1)$ currents $J_\mu(x)$, where $J_\mu(x)$ is the vector current of quark field or quark number current. Now we define the precise form of the retarded Green functions which we discuss in this paper:

$$G_{\mu \nu}(k) = -i \int d^4x e^{-ikx} \theta(t) \langle [J_\mu(x), J_\nu(0)] \rangle. \quad (\text{A}\cdot 5)$$

Appendix B

— Chiral susceptibility in D3/D7 system —

In this appendix, we consider the chiral susceptibility. Note that in D3/D7 system, we have only $U(1)$ axial symmetry. Chiral susceptibility χ_c is one of the important observables in terms of chiral symmetry restoration:

$$\chi_c \equiv \frac{\partial \langle \bar{\psi} \psi \rangle}{\partial M_q}. \quad (\text{B}\cdot 1)$$

In the D3/D7 system, the quark condensate $\langle \bar{\psi} \psi \rangle$ is given by (3.17). Note that our chiral condensate $\langle \bar{\psi} \psi \rangle$ is defined to be positive since c is negative, while it is negative in ordinary QCD. Reading out the numerical value of c , we calculate the chiral susceptibility.

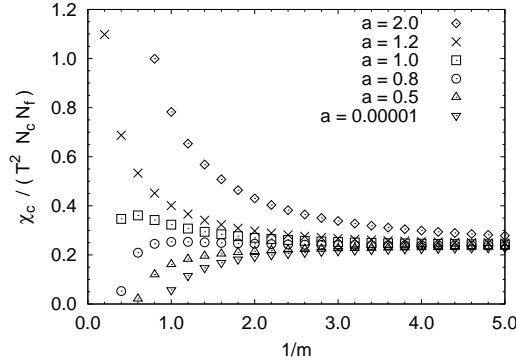


Fig. 3. The chiral susceptibility as function of $1/m$.

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